Asymmetrical Primitive Centrifuge Configurations and Vanishing Sums of Roots of Unity

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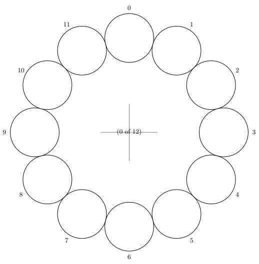
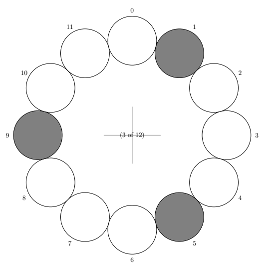
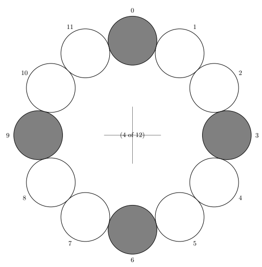
Introduction and Background

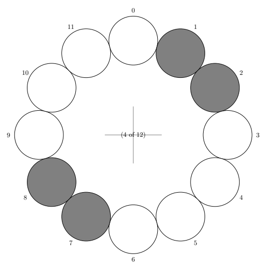
A centrifuge has some fixed number of positions into each of which a sample may be loaded. In this article, we consider only those with equally-spaced positions around a single circle. Real, commercial centrifuges come in a range of capacities, including with 4, 6, 8, 10 (rarely), 12, 16, 24, and (occasionally) 30 positions.



Equal-weight samples must be placed so that they're perfectly balanced.







The number of positions seems always to be even, and is often divisible by 6, to allow easy balancing. With $6k$ positions, any number of samples may be balanced, with the exceptions of 1 and $6k - 1$. For an even number, simply load in pairs into opposite positions. For an odd number, load 3 into positions in an equilateral triangle, and the rest into opposite pairs (from the $3(k - 1)$ such pairs that are still available).

If the number of samples can't be balanced in a centrifuge, then additional 'dummy' samples are used. In a centrifuge with an even number of positions, only one such dummy is ever required. When there are $6k$ positions, a dummy is only required for 1 and $6k - 1$ samples.

Notation

We will write

$$\{p_1, p_2, \ldots, p_k\}/n$$

for a **configuration** of k samples in positions p_1 through p_k in an n-position centrifuge. Usually, we will choose

$$0 \le p_1 < p_2 < \cdots < p_k < n.$$

This notation is intended to be suggestive of:

$$\{p_1, p_2, \ldots, p_k\}/n \;\sim\; \{p_1/n, p_2/n, \ldots, p_k/n\}.$$

We will also have that

$$\{\ldots, p_i, \ldots, p_j, \ldots\}/n \;\equiv\; \{\ldots, p_j, \ldots, p_i, \ldots\}/n;$$

$$\{p_1, \ldots, p_j, \ldots, p_k\}/n \;\equiv\; \{p_1, \ldots, p_j \pm n, \ldots, p_k\}/n.$$

Configurations such as

$$\{0, 4, 8\}/12$$

$$\{0, 2, 4\}/6$$

are clearly related. Our notation suggests

$$\{0, 4, 8\}/12 \;\sim\; \{0/12, 4/12, 8/12\} = \{0/6, 2/6, 4/6\} \;\sim\; \{0, 2, 4\}/6$$

but we consider these configurations distinct, and not equal, as the available or empty positions differ.

Our notation extends with some operators thus:

$$\begin{aligned} (\{p_1, p_2, \ldots, p_k\} + q)/n \quad &:= \quad \{p_1 + q, p_2 + q, \ldots, p_k + q\}/n \\ -\{p_1, p_2, \ldots, p_k\}/n \quad &:= \quad \{-p_k, -p_{k-1}, \ldots, -p_1\}/n \end{aligned}$$

Given a configuration

$$\{p_1, p_2, \ldots, p_k\}/n$$

we consider the clearly related (respectively by rotation and reflection) configurations

$$(\{p_1, p_2, \ldots, p_k\} + q)/n$$

$$-\{p_1, p_2, \ldots, p_k\}/n$$

generally to be distinct and not equal.

We may also extend the operator notation further in obvious ways:

$$\begin{aligned} (q + \{p_1, p_2, \ldots, p_k\})/n \quad &:= \quad (\{p_1, p_2, \ldots, p_k\} + q)/n \\ (\{p_1, p_2, \ldots, p_k\} - q)/n \quad &:= \quad (\{p_1, p_2, \ldots, p_k\} + (-q))/n \\ (q - \{p_1, p_2, \ldots, p_k\}/n \quad &:= \quad -(\{p_1, p_2, \ldots, p_k\} - q)/n \end{aligned}$$

Definitions of Properties

We might consider a configuration essentially to be a Boolean linear combination of roots of unity. In a configuration, each position may contain either zero or one sample.

The (complex) value of configuration

$$C = \{p_1, p_2, \ldots, p_k\}/n$$

is defined as

$$\|C\| = \sum_{j=1}^{k} \operatorname{cis}\left(\frac{p_j}{n}\tau\right)$$

where

$$\operatorname{cis}\vartheta = \cos\vartheta + \mathrm{i}\sin\vartheta = \mathrm{e}^{\mathrm{i}\,\vartheta}\,.$$

Balance

A configuration

$$C = \{p_1, p_2, \ldots, p_k\}/n$$

will be called **balanced** if

$$\|C\| = 0.$$

Otherwise it is **imbalanced**.

In this article we are interested only in balanced configurations.

Primitiveness

A balanced configuration will be called **primitive** if it is non-empty and no non-empty proper subset is balanced.

Primitive configurations are usually not full. These occur only if the number of positions is a prime number.

If a balanced configuration is not primitive, then it is called **compound**.

For example, the configurations

$$\{0, 4, 8\}/12$$
$$\{3, 9\}/12$$

are primitive, but their combination

$$\{0, 3, 4, 8, 9\}/12$$

is compound.

The configuration

$$\{0, 2, 4, 6, 8, 10\}/12$$

is also compound rather than primitive, as it has balanced non-empty proper subsets, including

$$\{2, 8\}/12,$$
$$\{0, 4, 8\}/12.$$

Symmetry

A configuration has **rotational symmetry** if for some $q \,|\, n$, $1 \leq q < n$

$$(\{p_1, p_2, \ldots, p_k\} + q)/n = \{p_1, p_2, \ldots, p_k\}/n;$$

if q is the smallest such value, then the order of the rotational symmetry is n/q.

A configuration has **reflection symmetry** if for some q

$$(q - \{p_1, p_2, \ldots, p_k\})/n = \{p_1, p_2, \ldots, p_k\}/n.$$

The line of symmetry might be considered as being through positions $\frac{q}{2}$ and/or $\frac{q+n}{2}$, or at angle $\frac{q}{2n}\tau$.

A configuration having at least one of these kinds of symmetry will be called **symmetric**.

A configuration having neither kind of symmetry will be called **asymmetric**.

Primeness

A configuration that is both primitive and symmetric is called **prime**.

Canonical Notation

We might consider that certain configurations are equivalent under simple transformations such as rotation or reflection.

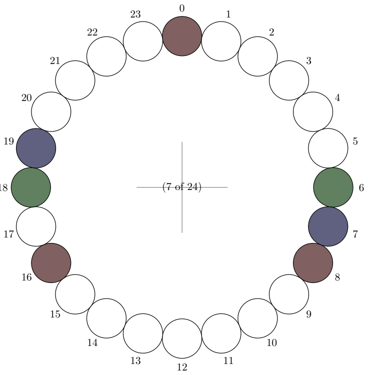
In order to distinguish inequivalent configurations, the 'canonical' configuration might be considered as representative of the class. This might be taken to be, for example, the lexicographically-minimal member of the class.

The canonical notation for the configuration might further have the sample values sorted, and numbered between 0 (inclusive) and the number of positions (exclusive).

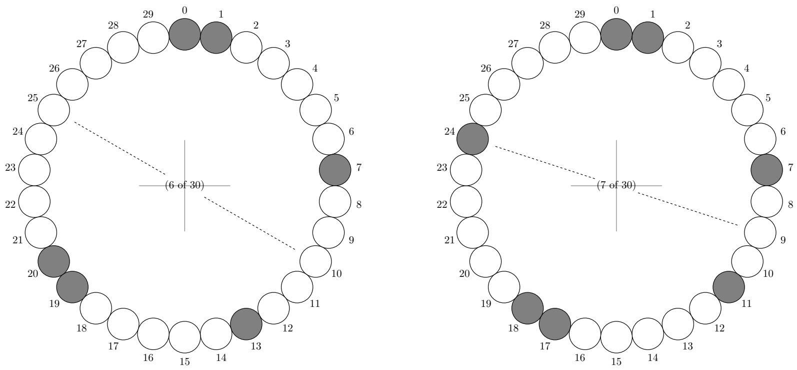
Symmetric Primitive Configurations

In some laboratories, lab technicians amuse themselves by finding balanced configurations that do not look balanced.

The examples I have seen online are all compound configurations, though usually asymmetric.



In seeking primitive configurations that do not look balanced (or are not prime), initially I found only examples having reflection symmetry.

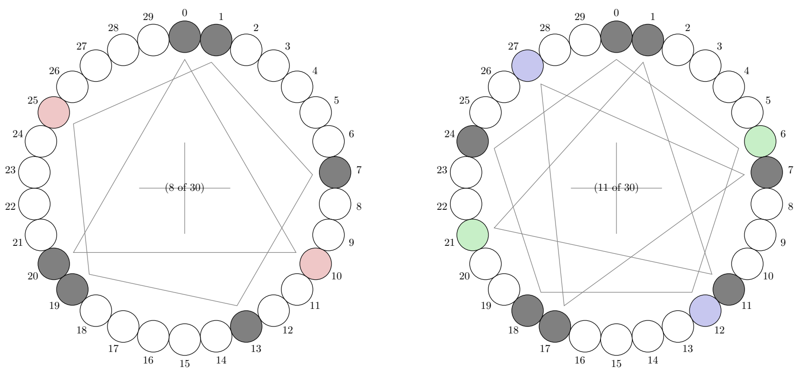


Here we have

$$\{0, 1, 7, 13, 19, 20\}/30$$
$$\{0, 1, 7, 11, 17, 18, 24\}/30$$

That these configurations are balanced can be seen geometrically.

Remove the prime opposing pairs (shown in pale colours) to produce these primitive configurations. See the later section of the composition of the asymmetric configuration for a more-detailed explanation.



These configurations are symmetric, and thus are not the main topic of this article.

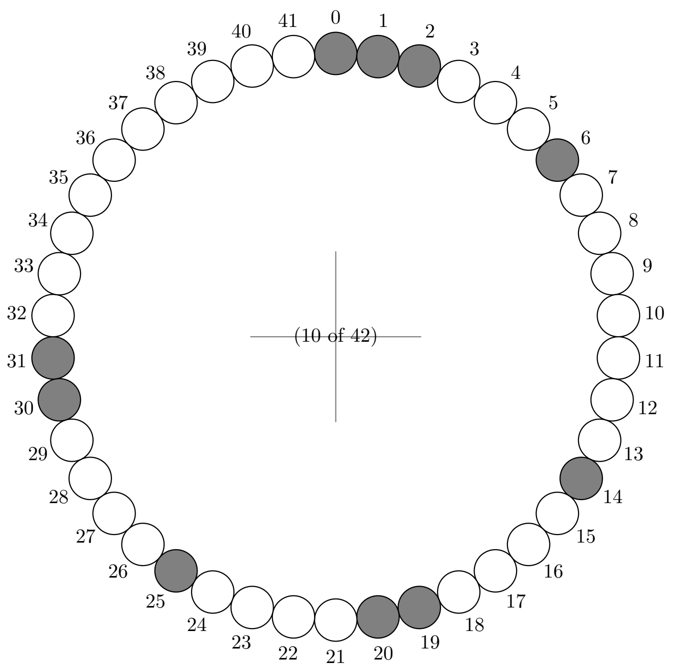
An Asymmetric Primitive Configuration

Initially, it seemed that an asymmetric primitive configuration might require $210 = 2 \times 3 \times 5 \times 7$ available positions. However, it turns out that there is a primitive configuration requiring $42 = 2 \times 3 \times 7$ positions.

Illustrated is such an asymmetric primitive configuration on 42 positions. The configuration

$$\{0, 1, 2, 6, 14, 19, 20, 25, 30, 31\}/42$$

with 10 samples, is balanced and in fact primitive. (This is the canonical notation for this configuration class.)



Sums of Roots of Unity

The sum of all the n-th roots of unity vanishes (for an integer $n \geq 2$).

$$\sum_{k=0}^{n-1} \xi^k = 0 \quad \text{where} \quad \xi = \xi_n = \operatorname{cis}\left(\frac{\tau}{n}\right)$$

where

$$\operatorname{cis}(\vartheta) = \cos\vartheta + \mathrm{i}\sin\vartheta = \exp(\mathrm{i}\,\vartheta) = \mathrm{e}^{\mathrm{i}\,\vartheta}$$

i.e. ξ is a primitive n-th root of unity, i.e., roughly,

$$\xi \sim 1^{1/n} \sim \sqrt[n]{1}.$$

Note that

$$\xi_n{}^t = \xi_{kn}{}^{kt}.$$

Balanced or Prime Sums of Roots of Unity

By definition, balanced sets of n-th roots of unity also vanish.

Prime sets of such roots are of some interest. For example:

$$\xi_{24}{}^{3} + \xi_{24}{}^{11} + \xi_{24}{}^{19} = 0$$

where

$$\xi_{24} = \operatorname{cis}\left(\frac{\tau}{24}\right) = \frac{(\sqrt{3}+1)\sqrt{2}}{4} + \mathrm{i}\,\frac{(\sqrt{3}-1)\sqrt{2}}{4}.$$

Expanding the above, we have

$$\left(\frac{\sqrt{2}}{2} + \mathrm{i}\,\frac{\sqrt{2}}{2}\right) + \left(-\frac{(\sqrt{3}+1)\sqrt{2}}{4} + \mathrm{i}\,\frac{(\sqrt{3}-1)\sqrt{2}}{4}\right) + \left(\frac{(\sqrt{3}-1)\sqrt{2}}{4} - \mathrm{i}\,\frac{(\sqrt{3}+1)\sqrt{2}}{4}\right) = 0.$$

Such sums are not particularly interesting or surprising.

Part-symmetrical Primitive Sums of Roots of Unity

Primitive sums with only reflection symmetry are somewhat interesting. The primitive configurations

$$\{0, 1, 7, 13, 19, 20\}/30$$

$$\{0, 1, 7, 11, 17, 18, 24\}/30$$

give rise to

$$\xi_{30}{}^{0} + \xi_{30}{}^{1} + \xi_{30}{}^{7} + \xi_{30}{}^{13} + \xi_{30}{}^{19} + \xi_{30}{}^{20} = 0,$$

$$\xi_{30}{}^{0} + \xi_{30}{}^{1} + \xi_{30}{}^{7} + \xi_{30}{}^{11} + \xi_{30}{}^{17} + \xi_{30}{}^{18} + \xi_{30}{}^{24} = 0.$$

The first of these may be demonstrated mathematically by:

$$\begin{aligned}
&\xi_{30}{}^{0} + \xi_{30}{}^{1} + \xi_{30}{}^{7} + \xi_{30}{}^{13} + \xi_{30}{}^{19} + \xi_{30}{}^{20} \\
&= (\xi_{30}{}^{0} + \xi_{30}{}^{10} + \xi_{30}{}^{20}) + (\xi_{30}{}^{1} + \xi_{30}{}^{7} + \xi_{30}{}^{13} + \xi_{30}{}^{19} + \xi_{30}{}^{25}) - (\xi_{30}{}^{10} + \xi_{30}{}^{25}) \\
&= (\xi_{30}{}^{0} + \xi_{30}{}^{10} + \xi_{30}{}^{20}) + (\xi_{30}{}^{0} + \xi_{30}{}^{6} + \xi_{30}{}^{12} + \xi_{30}{}^{18} + \xi_{30}{}^{24}) \cdot \xi_{30} - (\xi_{30}{}^{0} + \xi_{30}{}^{15}) \cdot \xi_{30}{}^{10} \\
&= (\xi_{3}{}^{0} + \xi_{3}{}^{1} + \xi_{3}{}^{2}) + (\xi_{5}{}^{0} + \xi_{5}{}^{1} + \xi_{5}{}^{2} + \xi_{5}{}^{3} + \xi_{5}{}^{4}) \cdot \xi_{30} - (\xi_{2}{}^{0} + \xi_{2}{}^{1}) \cdot \xi_{3} \\
&= 0 + 0 \cdot \xi_{30} - 0 \cdot \xi_{3} \\
&= 0.
\end{aligned}$$

The second may similarly be demonstrated.

Asymmetrical Primitive Sums of Roots of Unity

Asymmetrical primitive sums are more interesting.

Recall our asymmetrical primitive configuration

$$\{0, 1, 2, 6, 14, 19, 20, 25, 30, 31\}/42.$$

That this balances implies

$$\xi^0 + \xi^1 + \xi^2 + \xi^6 + \xi^{14} + \xi^{19} + \xi^{20} + \xi^{25} + \xi^{30} + \xi^{31} = 0$$

where

$$\xi = \xi_{42} = \operatorname{cis}\left(\frac{\tau}{42}\right) \sim \sqrt[42]{1}$$

i.e.

$$1 + \xi + \xi^2 + \xi^6 + \xi^{14} + \xi^{19} + \xi^{20} + \xi^{25} + \xi^{30} + \xi^{31} = 0.$$

Again as with part-symmetrical configurations, this may be demonstrated mathematically. However, we give a more-geometrical demonstration in the next section.

Given that $\xi^{42} = 1$ and $\xi^{21} = -1$, and, since $\xi^{42-k} = \overline{\xi^k}$, this is equivalent to

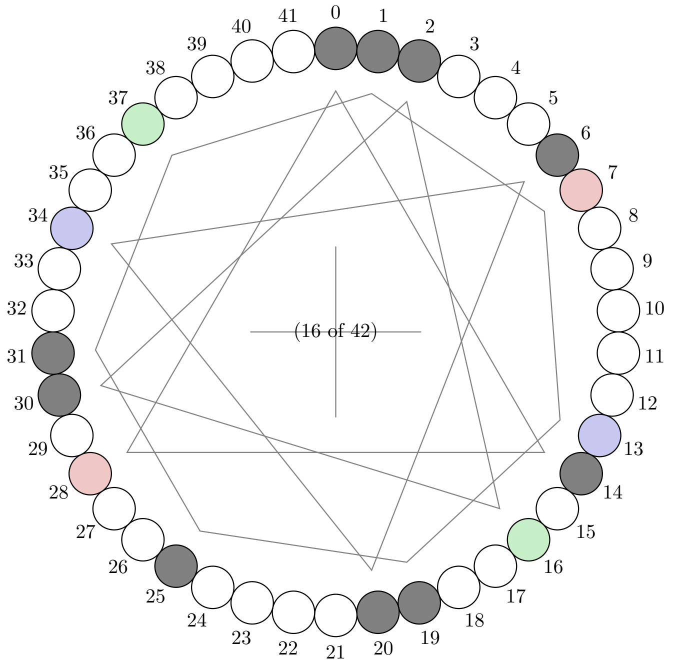
$$\xi^4 + \xi^9 + \xi^{10} + \xi^{22} + \xi^{23} + \xi^{27} + \xi^{35} + \xi^{40} + \xi^{41} = 1,$$

$$\xi + \xi^2 + \xi^7 + \xi^{15} + \xi^{19} + \xi^{20} + \xi^{32} + \xi^{33} + \xi^{38} = 1,$$

and many other similar, essentially equivalent, expressions by rotating (multiplying by ξ) and reflecting (taking complex conjugates) our balanced configurations.

The Composition of the Configuration

That the asymmetric configuration is balanced can be seen geometrically.



We begin with the prime configurations for a regular heptagon and three equilateral triangles

$$\begin{aligned}
\{1, 7, 13, 19, 25, 31, 37\}/42 &= (\{0, 6, 12, 18, 24, 30, 36\} + 1)/42 \\
\{0, 14, 28\}/42 & \\
\{2, 16, 30\}/42 &= (\{0, 14, 28\} + 2)/42 \\
\{6, 20, 34\}/42 &= (\{0, 14, 28\} + 6)/42
\end{aligned}$$

form the compound, and then remove the prime opposing pairs (shown in pale colours)

$$\begin{aligned}
\{7, 28\}/42 &= (\{0, 21\} + 7)/42 \\
\{13, 34\}/42 &= (\{0, 21\} + 13)/42 \\
\{16, 37\}/42 &= (\{0, 21\} + 16)/42
\end{aligned}$$

That the result is primitive may be checked by brute force.