On the Visualisation of Inequalities amongst Homogeneous Means

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This article presents a visual method of comparing means, and of gaining an overview of the behaviour over the entire domain.

Domain and Codomain: Binary Means on Non-negative Real Numbers

In this article, only means on either the positive real numbers

$$M : R^n \to R \quad \text{for } n \in \mathbb{N}^+$$

$$R = \mathbb{R}^+ = \mathbb{R}(0, \infty)$$

or on the non-negative reals with infinity

$$R = \mathbb{R}[0, \infty] = \mathbb{R}_0^+ \cup \{\infty\} = \mathbb{R}[0, \infty) \cup \{\infty\}$$

are considered.

Further, we consider only binary means

$$M : R \times R \to R$$

Thus, we consider binary means

$$M : \mathbb{R}(0, \infty)^2 \to \mathbb{R}(0, \infty)$$

or

$$M : \mathbb{R}[0, \infty]^2 \to \mathbb{R}[0, \infty]$$

Axioms

Axioms are the properties that must be satisfied by a function for it to qualify as a mean. When considering a general or arbitrary mean, the axioms are the properties that are assumed.

Some means might have additional properties.

In this article, the axioms of a mean are taken to be:

(Int) Internality

The mean of a collection of values must lie between the extrema.

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$$

or

$$x \leq y \implies x \leq M(x, y) \leq y$$

This implies in particular, reflexivity:

$$M(z, z) = z$$

Usually, our means will also satisfy *strict* internality

$$x \neq y \implies \min\{x, y\} < M(x, y) < \max\{x, y\}$$

or

$$x < y \implies x < M(x, y) < y$$

but we do not require this.

(Ref) Reflexivity

The mean of a collection containing only one value must be that value

$$M(z, z) = z$$

(Hom) Homogeneity

A homogeneous function is agnostic about scale.

$$M(cx, cy) = cM(x, y)$$

Note that homogeneity is usually defined for functions with

$$f(z_1, z_2, \ldots, cz_i \ldots, z_n) = cf(z_1, z_2, \ldots, z_n)$$

thus

$$f(cz_1, cz_2, \ldots, cz_n) = c^n f(z_1, z_2, \ldots, z_n)$$

but a different definition is applicable for means, sometimes called first-order homogeneity.

(Sym) Symmetry

$$M(x, y) = M(y, x)$$

(Con) Continuity

Continuity is an analytical attribute, unlike the other axioms, which are algebraic.
A mean is continuous in each argument

$$\lim_{\varepsilon \to 0} M(x + \varepsilon, y) = M(x, y)$$

$$\lim_{\varepsilon \to 0} M(x, y + \varepsilon) = M(x, y)$$

Order of Means

Given means G and H, we will write

$$H \leq G \quad \text{iff} \quad \forall\ x, y \bullet H(x,y) \leq G(x,y)$$

It is possible for a pair of means to be incomparable, i.e. not to be ordered.

We will also write, in the usual way

$$G \geq H \quad \text{iff} \quad H \leq G$$

$$H < G \quad \text{iff} \quad H \leq G\ \wedge\ H \neq G$$

Note that $H < G$ does **not** imply that $\forall\ x, y \bullet H(x,y) < G(x,y)$;
only that $\forall\ x, y \bullet H(x,y) \leq G(x,y)$ and $\exists\ x, y \bullet H(x,y) < G(x,y)$.

Distillation

Given a binary mean

$$M : \mathbb{R}[0,\infty]^2 \rightarrow \mathbb{R}[0,\infty]$$

we 'distil' this to a unary function on $\mathbb{R}[0,2]$

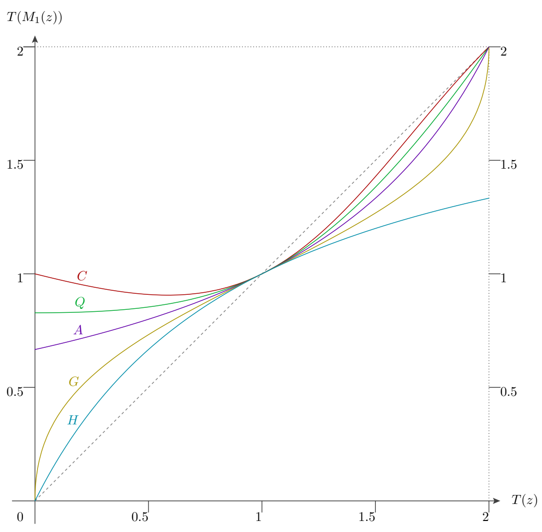
$$T(M_1) : \mathbb{R}[0,2] \rightarrow \mathbb{R}[0,2]$$

that still contains the 'essence' of M. Distillation comprises two parts: 'reduction' and 'compression' (described below).

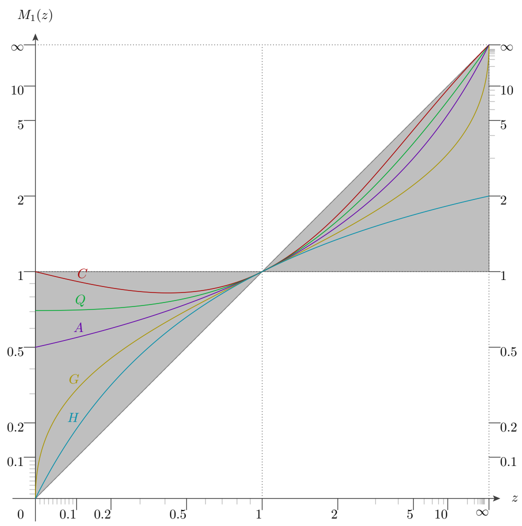
Specifically, given $T(M_1)$, the function M can be 'recovered'.

The (compression) transformation T is only to allow the (reduced) unary function M_1 to be plotted. The illustrations in this article use T for plotting, but show the unary function M_1 with a non-linear scale.

That is, although compression is used to create plots such as



the illustrations will be shown with the inverse of the compression function applied to the axes



so that we may regard the illustrations to be of reduced means.

Means should have the internality property, and thus should be contained in the shaded region.

The upper boundary of the shaded region represents the $\max(\dots)$ function, and the lower boundary represents the $\min(\dots)$ function.

Means should normally be monotonic. This diagram provides the insight that the contra-harmonic mean C is non-monotonic,

Reduction of Homgeneous Mean from Binary to Unary

Since we have assumed that means satisfy the homogeneity axiom, we may reduce our binary means to a unary function.

$$M_1(z) = M(1, z)$$

(The '1' subscript may be taken to represent that there is only a single argument, or to represent that the fixed argument has value 1.)

This function essentially collapse a pair of values to their ratio. So this transformation might also be called 'rationalisation'.

The binary mean may be recovered from the unary function.

$$M(x, y) = M_1(y/x) \cdot x$$

Compression of Unary Mean to Finite Domain

Define

$$T : \mathbb{R}[0, \infty] \to \mathbb{R}[0, 2]$$

$$z \mapsto \frac{2z}{z+1} \quad \text{for } 0 \leq z < \infty$$
$$\infty \mapsto 2$$

T is bijective, and thus we may define the inverse

$$T^{-1} : \mathbb{R}[0,2] \to \mathbb{R}[0,\infty]$$
$$z \mapsto \frac{z}{2-z} \quad \text{for } 0 \leq z < 2$$
$$2 \mapsto \infty$$

We have:

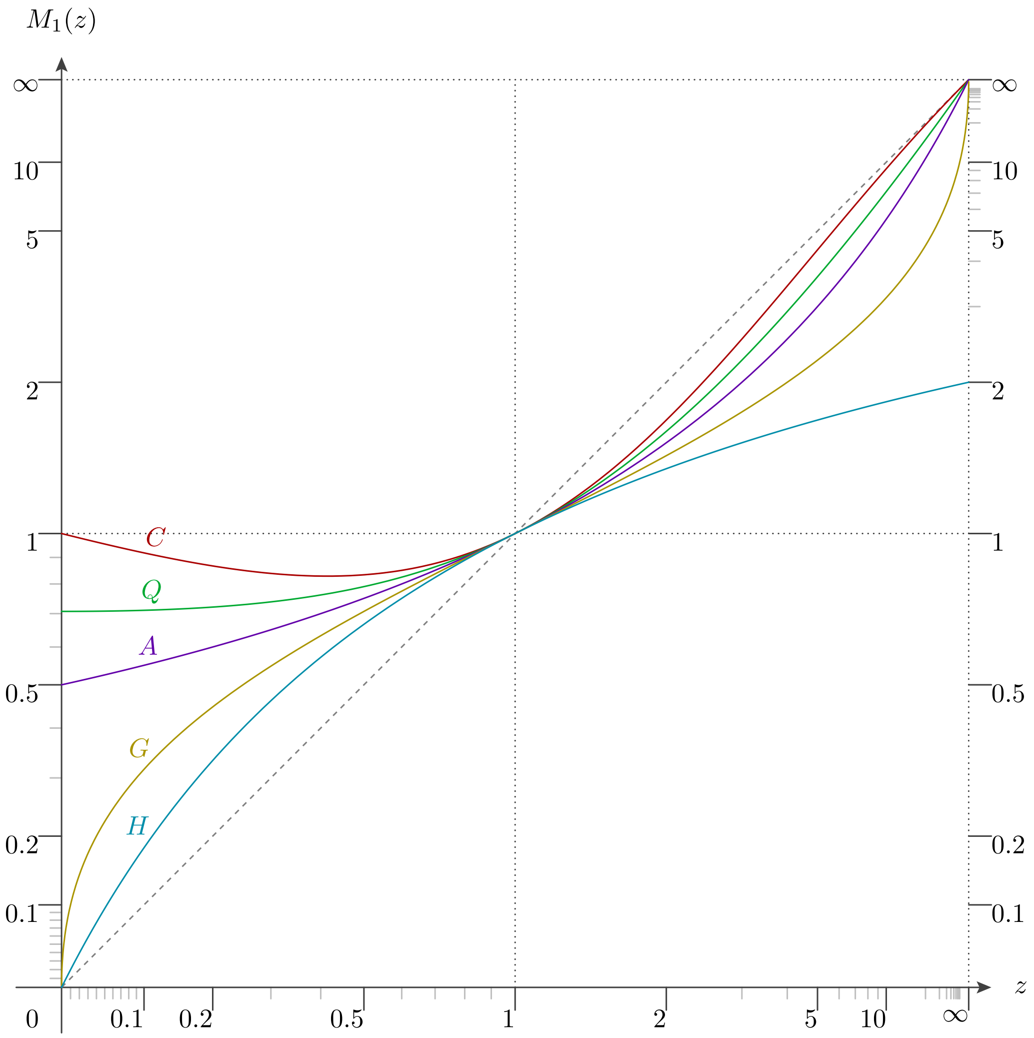
$$T(0) = 0$$
$$T(1) = 1$$
$$T(\infty) = 2$$

The transformation T is monotonically increasing:

$$x \leq y \implies T(x) \leq T(y)$$

Example Illustrations

The Pythagorean and Classical Means



The Pythagorean means are:

A : arithmetic mean $A(x, y) = \frac{x+y}{2}$
G : geometric mean $G(x, y) = \sqrt{xy}$
H : harmonic mean $H(x, y) = \frac{2xy}{x+y}$

The classical means also include:

C : contraharmonic mean $C(x, y) = \frac{x^2+y^2}{x+y}$

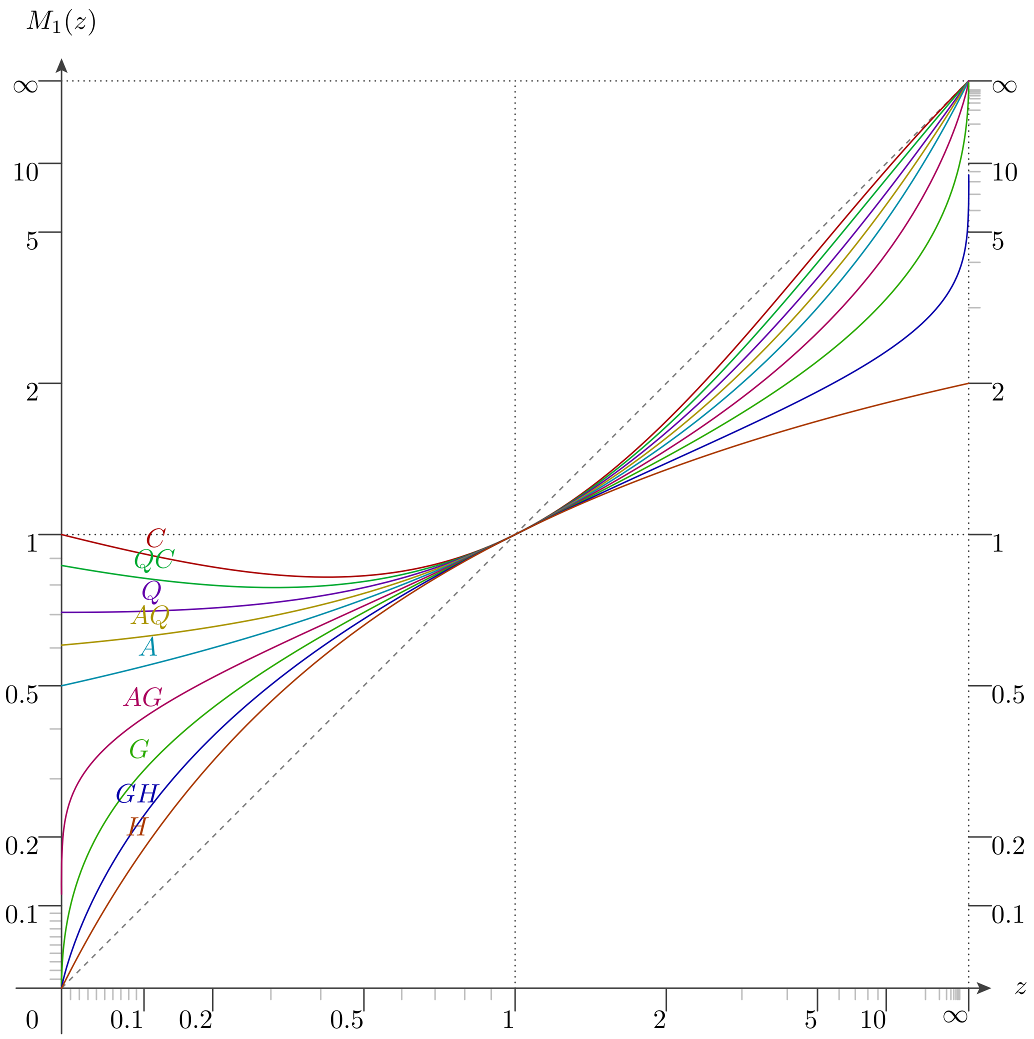
Q : quadratic mean (aka root mean square, RMS) $Q(x, y) = \sqrt{\frac{x^2+y^2}{2}}$

Notice that these curves are all monotonic except for C. This diagram provides the insight that the binary mean C is non-monotonic, and thus less well behaved than the other classical means.

We have

$$\min \leq H \leq G \leq L \leq A \leq Q \leq C \leq \max$$

where L is defined below.



The iterative means include:

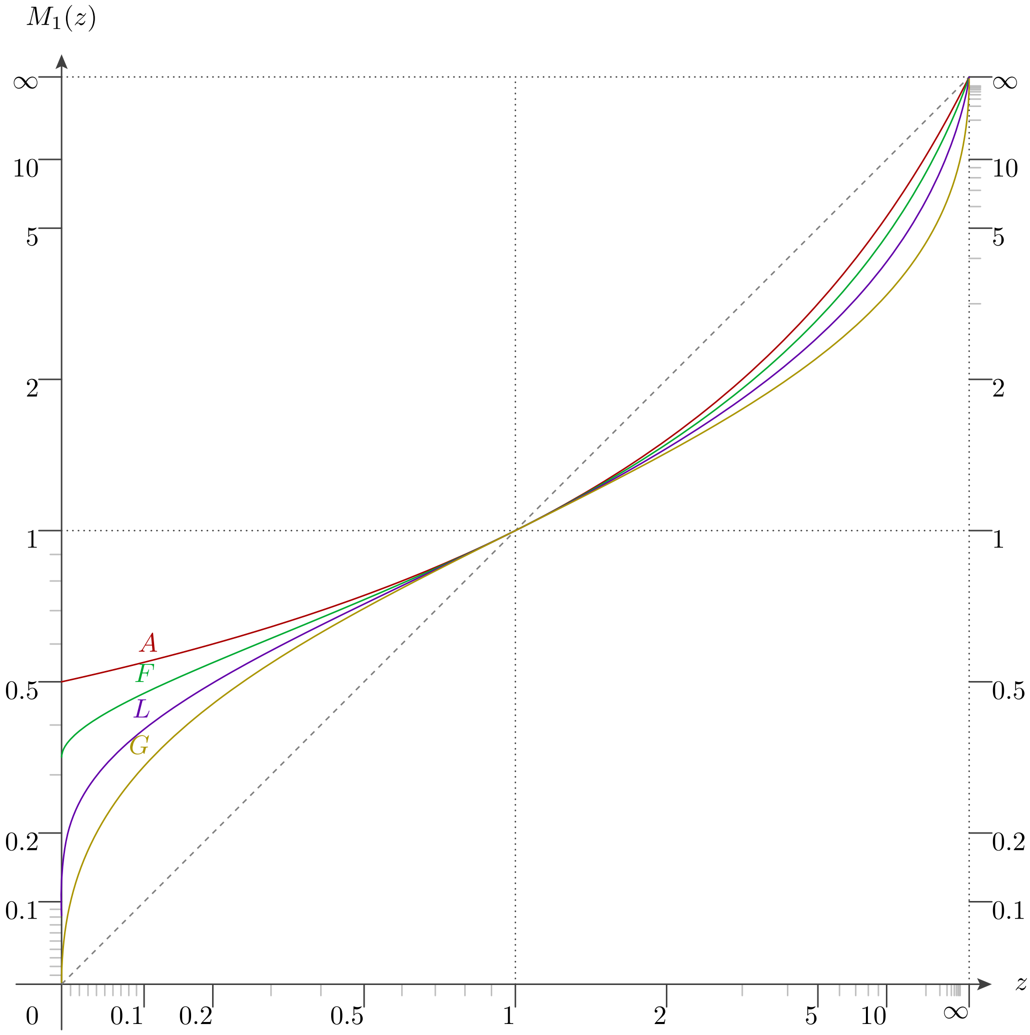
AG : arithmetic–geometric mean
GH : geometric–harmonic mean

The iterative means are ordered between those being combined and iterated.

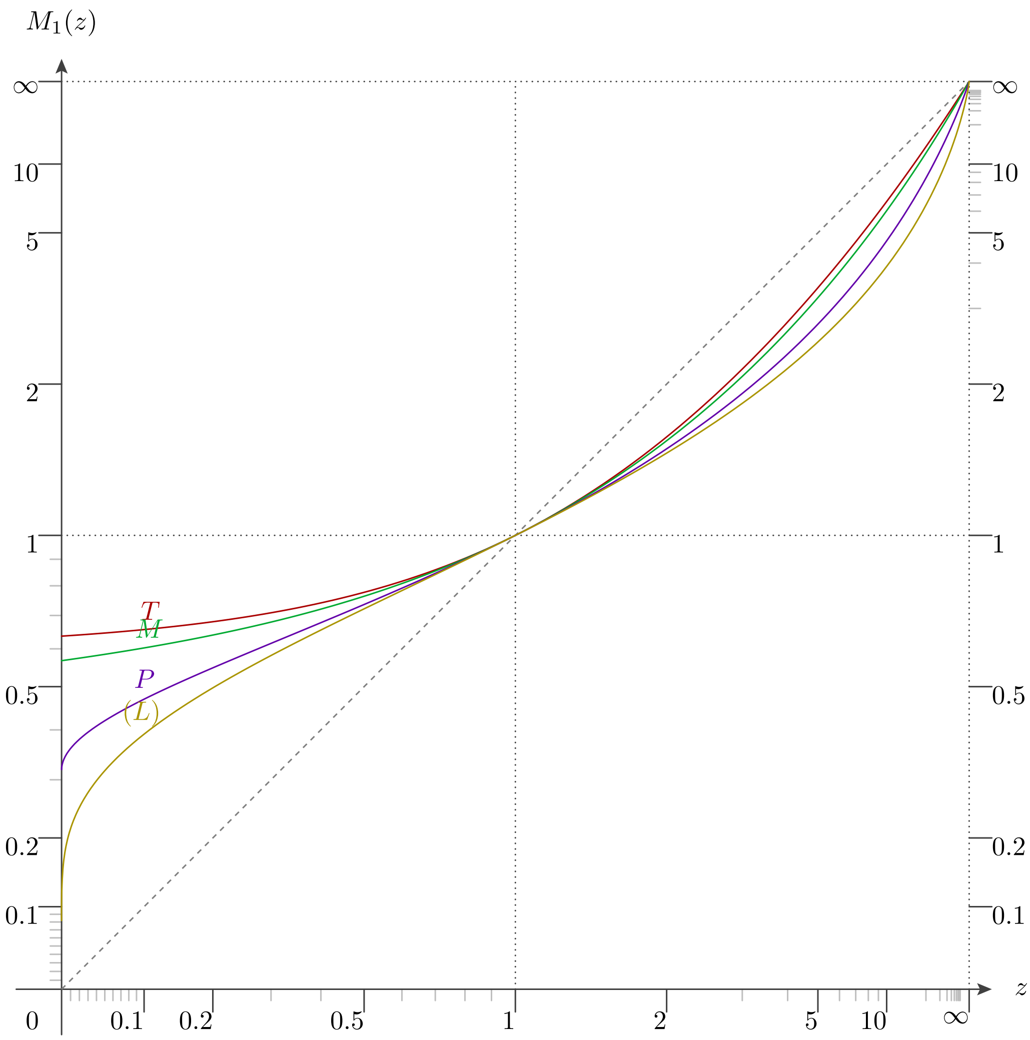
We have, for example

$$H \leq GH \leq G \leq AG \leq A$$

Some Additional Means



F : Heronian mean $F(x,y) = \frac{1}{3}(x + \sqrt{xy} + y)$
L : logarithmic mean $L(x,y) = (x-y)/(\ln x - \ln y)$
I : identric mean $I(x,y) = \frac{1}{\mathrm{e}} \sqrt[(x-y)]{x^x/y^y}$
(not shown; it is close to F)



T : second Seiffert mean $T(x,y) = (x-y)/(2\tan^{-1}\frac{x-y}{x+y}))$

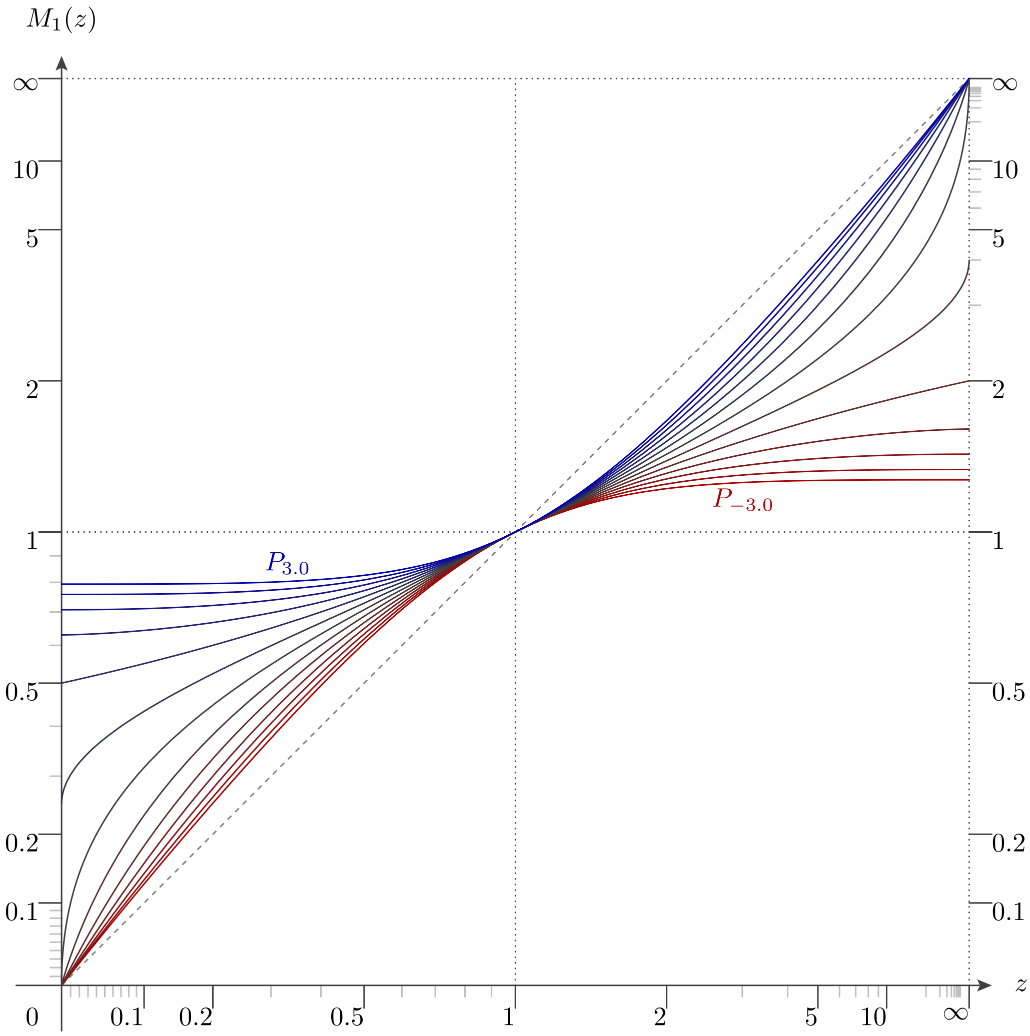
M : Neuman–Sándor mean $M(x,y) = (x-y)/(2\sinh^{-1}\frac{x-y}{x+y})$

P : first Seiffert mean $P(x,y) = (x-y)/(2\sin^{-1}\frac{x-y}{x+y})$

L : logarithmic mean (again); also $L(x,y) = (x-y)/(2\tanh^{-1}\frac{x-y}{x+y})$

The Power Means

Also known as Hölder means or generalised means.



$$P_k(x, y) = \sqrt[k]{\frac{x^k + y^k}{2}}$$

$$P_\infty = \max \; ; \quad P_2 = Q \; ; \quad P_1 = A \; ; \quad P_0 = G \; ; \quad P_{-1} = H \; ; \quad P_{-\infty} = \min$$

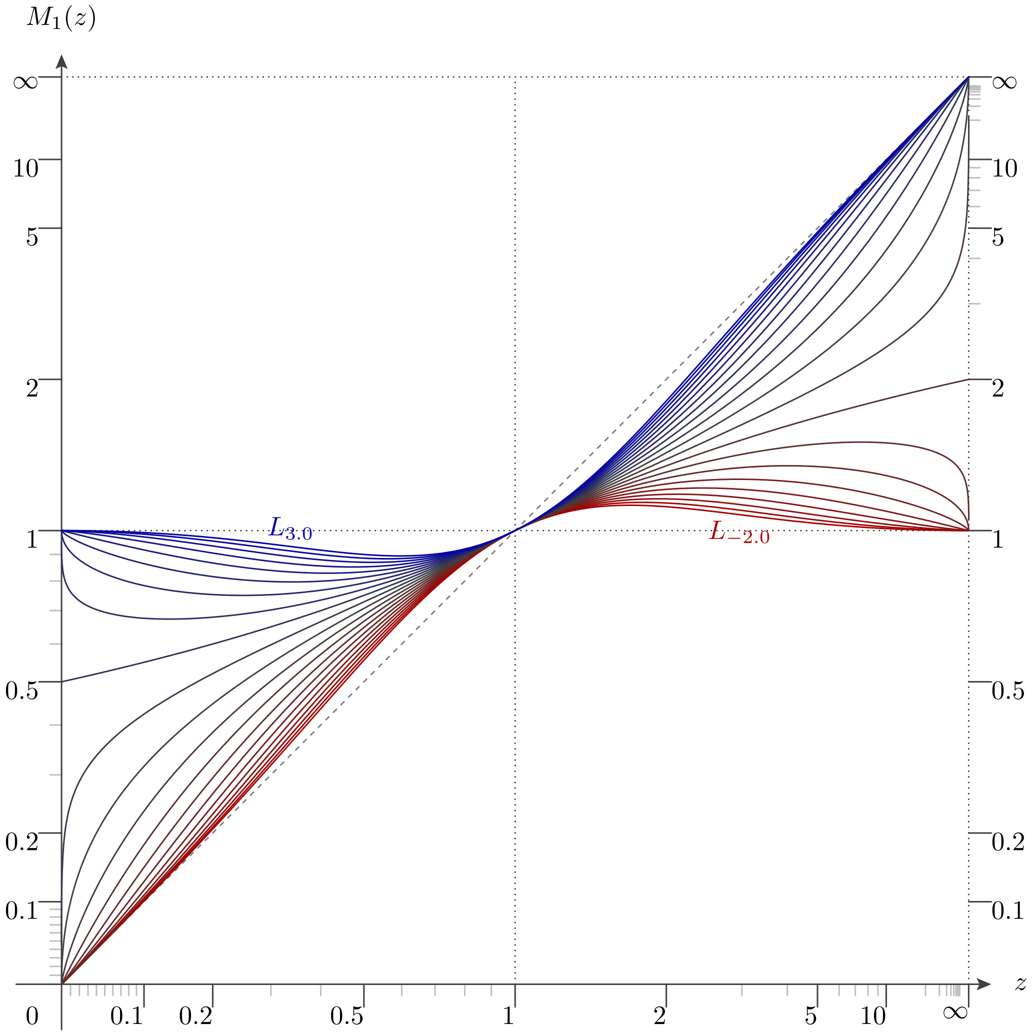
$$h \leq k \implies P_h \leq P_k$$

Strictly, the definition should be

$$P_k(x, y) = \lim_{h \to k} \sqrt[h]{\frac{x^h + y^h}{2}},$$

and evaluation of some cases requires L'Hôpital's rule and the manipulation of indeterminate forms, but I will omit limits for the sake of simplicity, as the main purpose of this article is to present the illustration of means using distillation.

The Lehmer Means

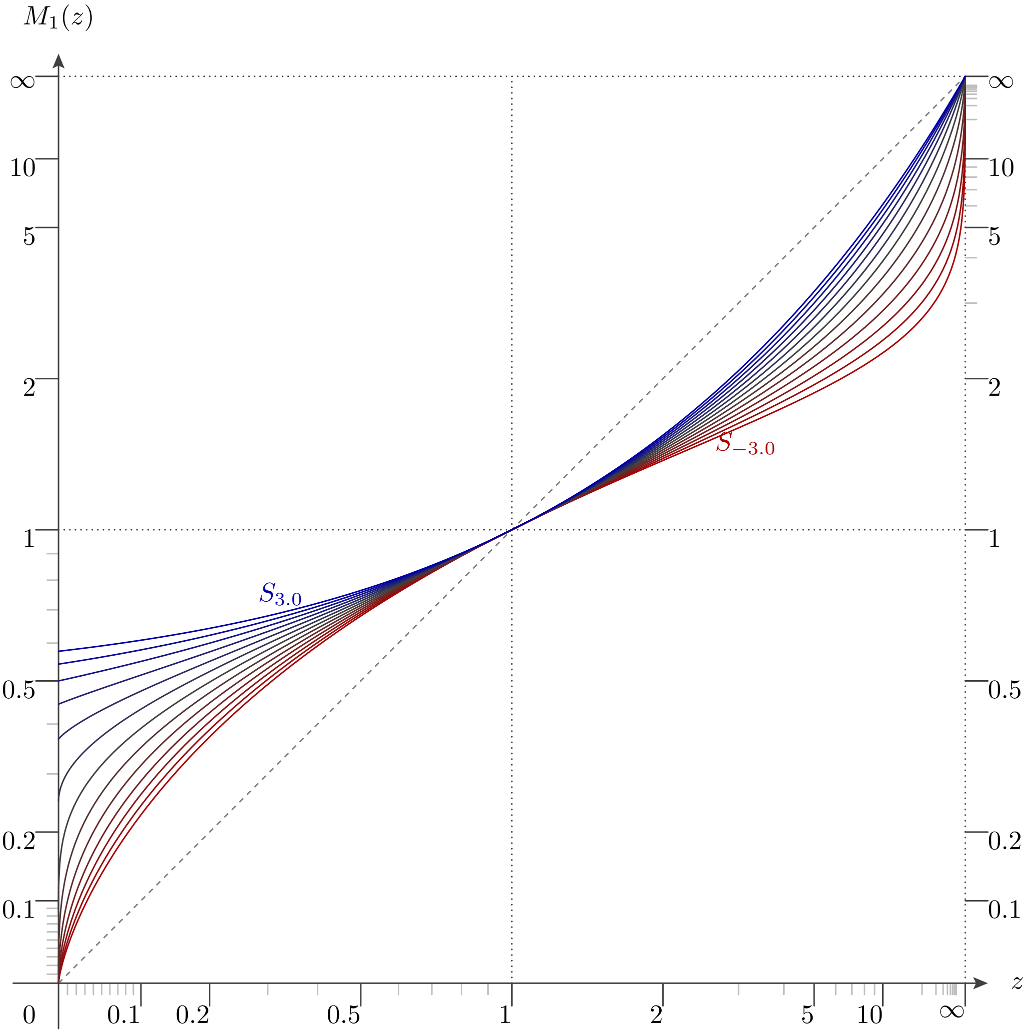


$$L_k(x,y) = \frac{x^k + y^k}{x^{k-1} + y^{k-1}}$$

$$L_\infty = \max \,; \quad L_2 = C \,; \quad L_1 = A \,; \quad L_{0.5} = G \,; \quad L_0 = H \,; \quad L_{-\infty} = \min$$

$$h \leq k \implies L_h \leq L_k$$

The Stolarsky Means

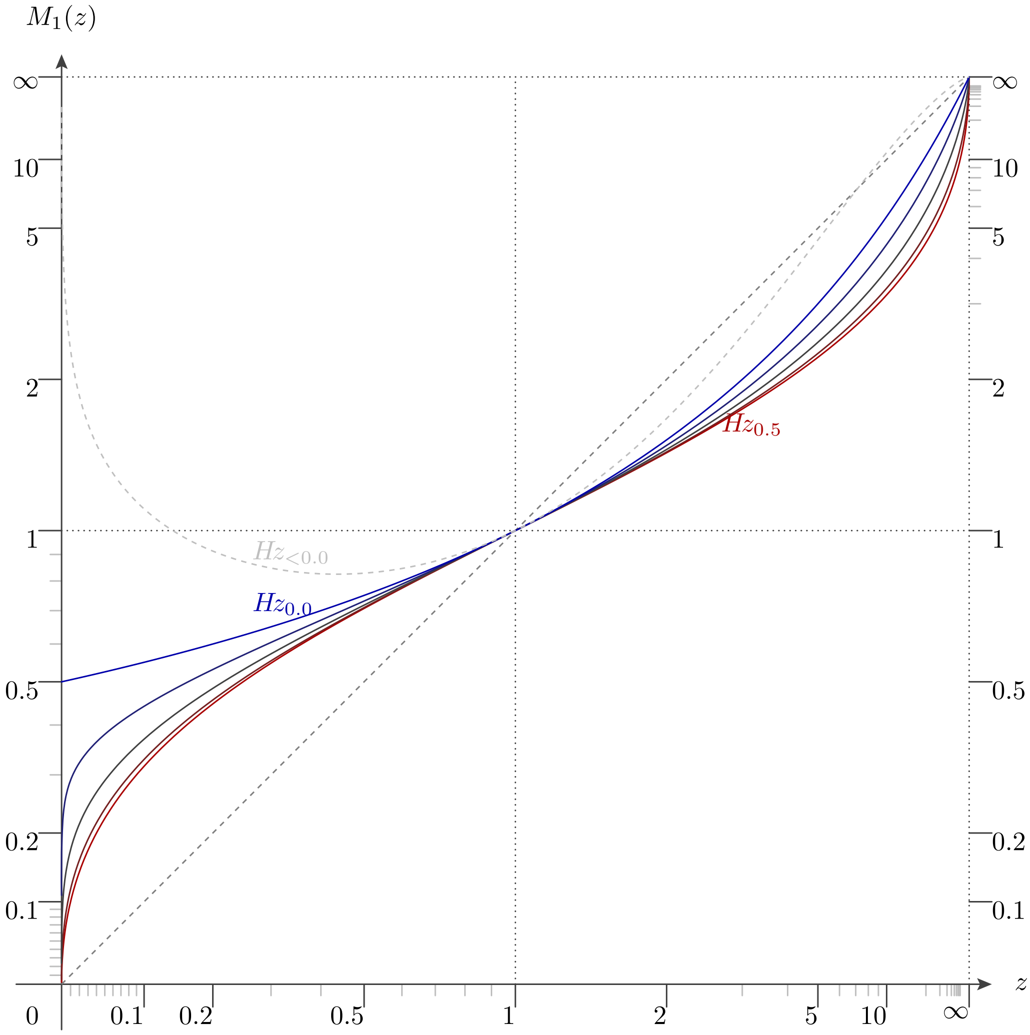


$$S_k(x,y) = \sqrt[k-1]{\frac{x^k - y^k}{k(x-y)}}$$

$$S_\infty = \max\,; \quad S_2 = A\,; \quad S_1 = I\,; \quad S_{0.5} = P_{0.5}\,; \quad S_0 = L\,; \quad S_{-1} = G\,; \quad S_{-\infty} = \min$$

$$h \leq k \implies S_h \leq S_k$$

The Heinz Means



$$Hz_k(x, y) = \frac{x^k y^{1-k} + x^{1-k} y^k}{2}$$

$$Hz_0 = A\ ; \quad Hz_{0.5} = G$$

Note that the parameter k is limited to $0 \le k \le 1/2$. The values $1/2 \le k \le 1$ are the same as for $1 - k$.

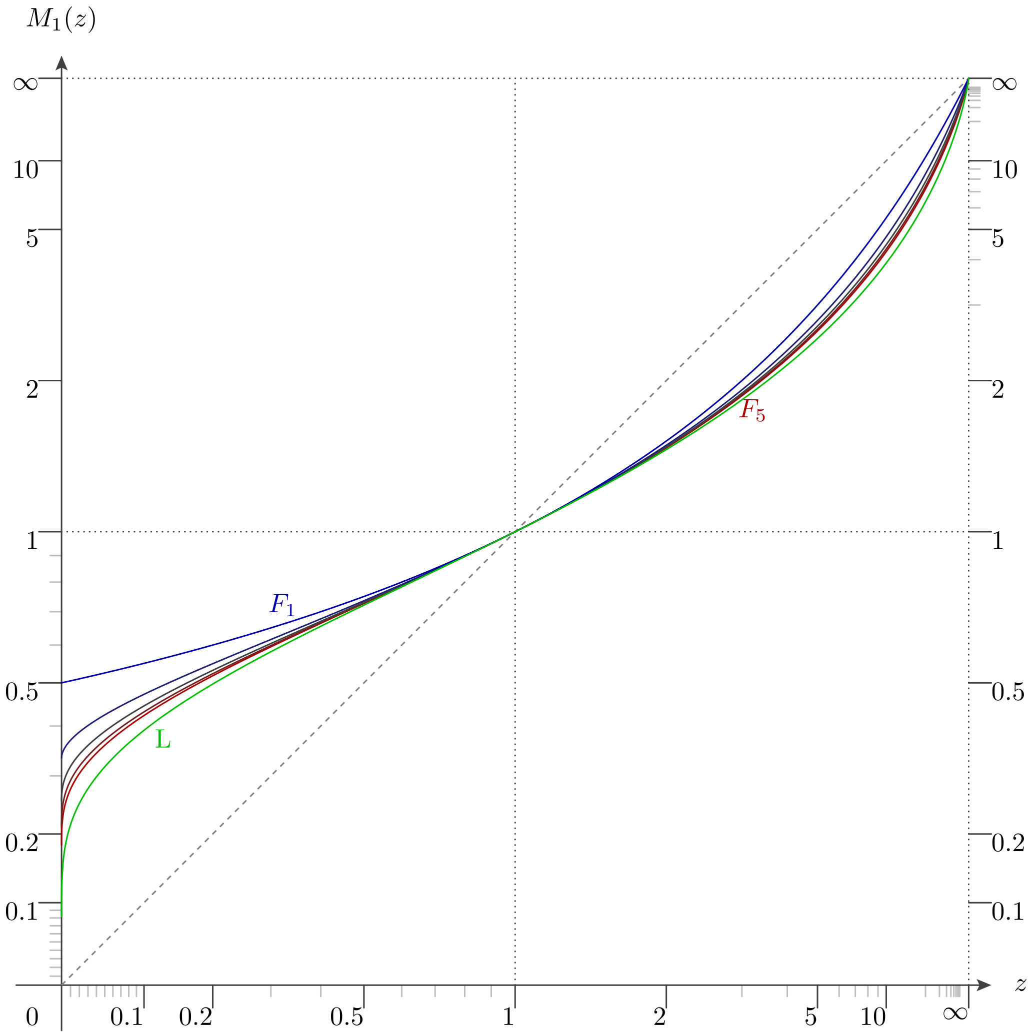
$$0 \le h \le k \le 1/2 \implies Hz_h \ge Hz_k$$

$$(1/2 \le h \le k \le 1 \implies Hz_h \le Hz_k)$$

Values $k < 0$ and $k > 1$ do not result in functions with internality, and thus those are not considered to be means. An example is illustrated with a dashed line.

Generalised Heronian Means

(These will be the subject of another article.)

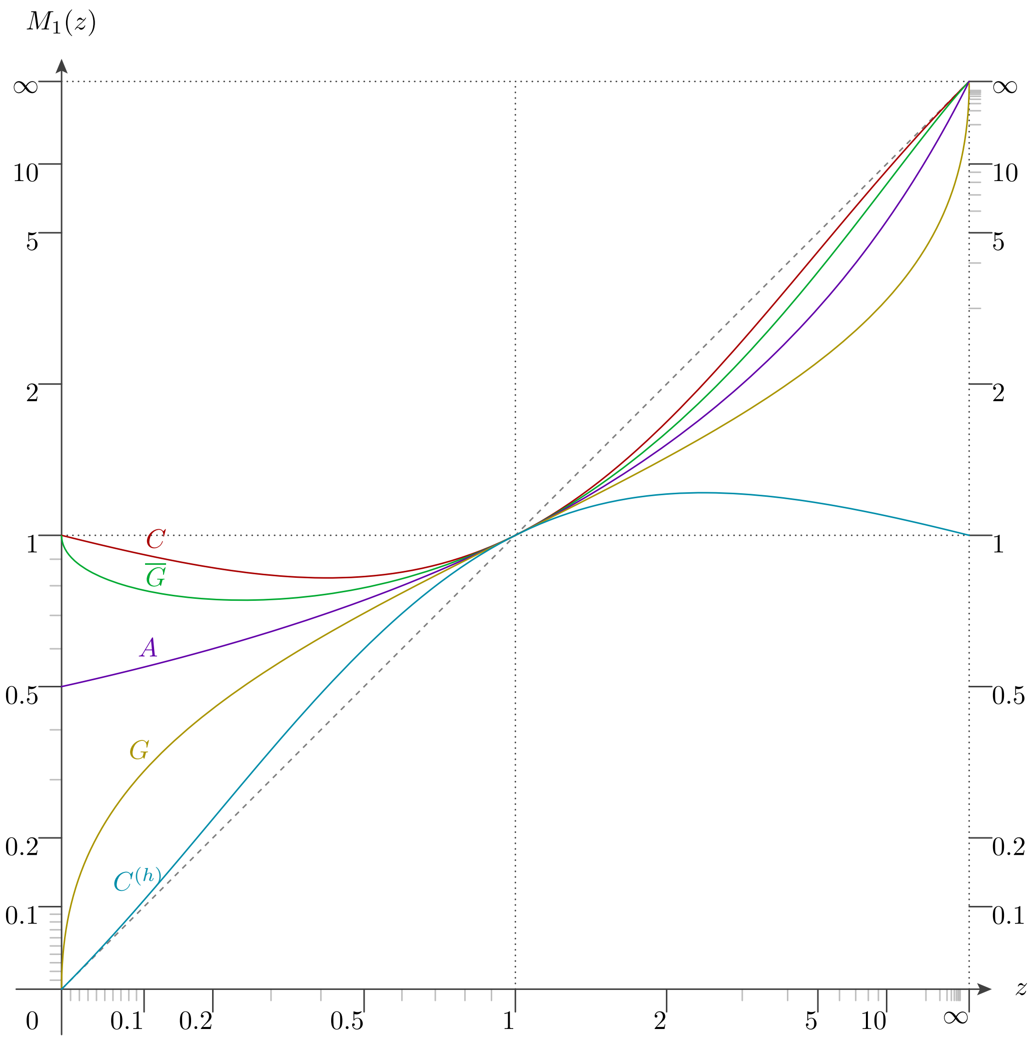


$$F_k(x,y) = \frac{x^{1-1/k} - y^{1-1/k}}{(k+1)(x^{1/k} - y^{1/k})}$$

$$F_1 = A\,; \quad F_2 = F\,; \quad F_\infty = L$$

Some Non-standard Means

These means are unlikely to be found in the literature. They are shown with some classical means for reference.



Here,

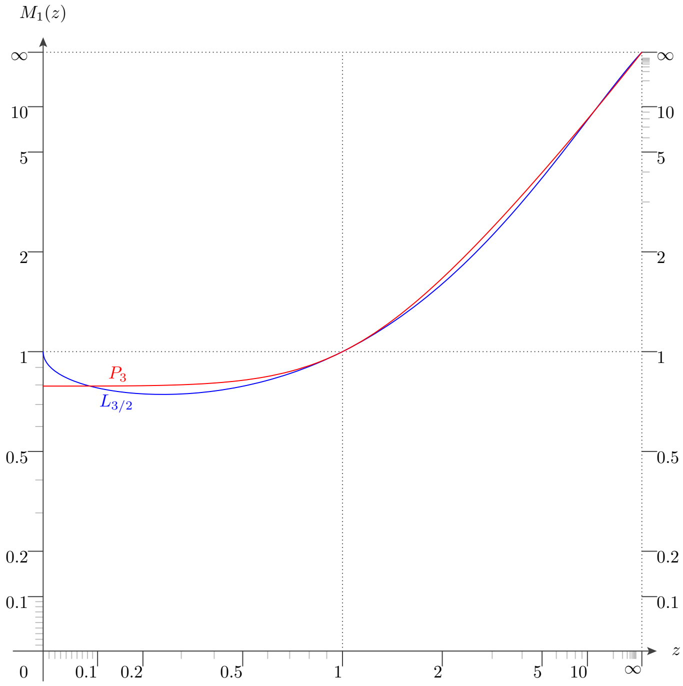
$$\overline{G}(x,y) = x + y - \sqrt{xy}$$

$$C^{(h)}(x,y) = \frac{xy(x+y)}{x^2+y^2}$$

We have $\overline{G} = 2A - G$ (akin to $C = \overline{H} = 2A - H$),
and $C^{(h)}(x,y) = 1/C(1/x, 1/y)$ (akin to $H = A^{(h)}$ and $G = G^{(h)}$).

Applications

Example Incomparable Means



Example Reflected Means

Means appear to be reflected (rather, rotated) when one is the harmonic of the other, i.e. when they are respectively in reciprocal spaces. The geomeric mean is its own harmonic.