Frustum Formulae

Rob Hubbard

We generalise the Heronian mean, and discover that these behave as a logarithmic mean in the limit.

3D

The formula for the volume of a frustum is

$$V = h \times \frac{A + \sqrt{AZ} + Z}{3}$$

where h is the perpendicular height, and A and Z are the areas of the end faces.

That is,

$$V = h \times M$$

where

$$M = \frac{A + \sqrt{AZ} + Z}{3}$$

is the mean cross sectional area. The mean area is the Heronian mean of the end areas.

Derivation

The linear (or length) dimension is in linear proportion to the position. The linear dimension is therefore proportional to the square root of the cross-sectional area.

Thus, taking a continuous mean over the height of the shape,

$$M = \frac{1}{\sqrt{Z} - \sqrt{A}} \int_{\sqrt{A}}^{\sqrt{Z}} x^2 \, \mathrm{d}x$$

For simplicity, let

$$a = \sqrt{A}$$
$$z = \sqrt{Z}$$

$$
\begin{aligned}
M &= \frac{1}{\sqrt{Z}-\sqrt{A}} \int_{\sqrt{A}}^{\sqrt{Z}} x^2 \, \mathrm{d}x \\
&= \frac{1}{z-a} \int_a^z x^2 \, \mathrm{d}x \\
&= \frac{1}{z-a} \left[\frac{x^3}{3} \right]_a^z \\
&= \frac{1}{z-a} \cdot \frac{z^3-a^3}{3} \\
&= \frac{1}{z-a} \cdot \frac{(z-a)\cdot(a^2+az+z^2)}{3} \\
&= \frac{a^2+az+z^2}{3} \\
&= \frac{A+\sqrt{AZ}+Z}{3}
\end{aligned}
$$

You might note from the above working that we also have the forms

$$
\begin{aligned}
M &= \frac{z^3-a^3}{3(z-a)} \\
&= \frac{\sqrt{Z}^3-\sqrt{A}^3}{3(\sqrt{Z}-\sqrt{A})} \\
&= \frac{\sqrt{Z^3}-\sqrt{A^3}}{3(\sqrt{Z}-\sqrt{A})} \\
&= \frac{Z^{\frac{3}{2}}-A^{\frac{3}{2}}}{3(Z^{\frac{1}{2}}-A^{\frac{1}{2}})}
\end{aligned}
$$

Generalisation

We can generalise this result for higher dimensions. Consider the $(n+1)$-dimensional case with n-dimensional 'ends'.

Again, for simplicity, let

$$
\begin{aligned}
a &= \sqrt[n]{A} \\
z &= \sqrt[n]{Z}
\end{aligned}
$$

$$
\begin{aligned}
M = M_n &= \frac{1}{\sqrt[n]{Z} - \sqrt[n]{A}} \int_{\sqrt[n]{A}}^{\sqrt[n]{Z}} x^n \, \mathrm{d}x \\
&= \frac{1}{z-a} \int_a^z x^n \, \mathrm{d}x \\
&= \frac{1}{z-a} \left[\frac{x^{n+1}}{n+1} \right]_a^z \\
&= \frac{1}{z-a} \cdot \frac{z^{n+1} - a^{n+1}}{n+1} \\
&= \frac{1}{n+1} \cdot \frac{1}{z-a} \cdot \left(z^{n+1} - a^{n+1} \right) \\
&= \frac{1}{n+1} \cdot \frac{1}{z-a} \cdot \left((z-a) \cdot \sum_{k=0}^{n} a^k z^{n-k} \right) \\
&= \frac{1}{n+1} \cdot \sum_{k=0}^{n} a^k z^{n-k} \\
&= \frac{1}{n+1} \cdot \sum_{k=0}^{n} A^{\frac{k}{n}} Z^{\frac{n-k}{n}} = \frac{1}{n+1} \cdot \sum_{k=0}^{n} A^{\frac{k}{n}} Z^{1-\frac{k}{n}} \\
&= \frac{1}{n+1} \cdot \sum_{k=0}^{n} \sqrt[n]{A^k Z^{n-k}}
\end{aligned}
$$

From the above, we also have the $\sum$-free form

$$
M = \frac{1}{n+1} \cdot \frac{\sqrt[n]{Z^{n+1}} - \sqrt[n]{A^{n+1}}}{\sqrt[n]{Z} - \sqrt[n]{A}}
$$

2D

We have $n = 1$.

$$
\begin{aligned}
M &= \frac{1}{2} \cdot \sum_{k=0}^{1} A^k Z^{1-k} \\
&= \frac{1}{2} \cdot \left(A^1 Z^0 + A^0 Z^1 \right) \\
&= \frac{A+Z}{2}
\end{aligned}
$$

as expected, where A and Z are the lengths of the ends. The mean length is just the (arithmetic) mean of the end lengths.

4D

We have $n = 3$.

Similarly to the 2D and 3D cases, we have

$$
\begin{aligned}
M &= \frac{1}{4} \cdot \sum_{k=0}^{3} A^{\frac{k}{3}} Z^{1-\frac{k}{3}} \\
&= \frac{1}{4} \cdot \left(\sqrt[3]{A^3 Z^0} + \sqrt[3]{A^2 Z^1} + \sqrt[3]{A^1 Z^2} + \sqrt[3]{A^0 Z^3} \right) \\
&= \frac{A + \sqrt[3]{A^2 Z} + \sqrt[3]{A Z^2} + Z}{4}
\end{aligned}
$$

where now, A and Z are the volumes of the 'ends'.

5D

We have $n = 4$.

Similarly to the 2D, 3D and 4D cases, we have

$$
\begin{aligned}
M &= \frac{1}{5} \cdot \sum_{k=0}^{4} A^{\frac{k}{4}} Z^{1-\frac{k}{4}} \\
&= \frac{1}{5} \cdot \left(A + \sqrt[4]{A^3 Z} + \sqrt[4]{A^2 Z^2} + \sqrt[4]{A Z^2} + Z \right) \\
&= \frac{A + \sqrt[4]{A^3 Z} + \sqrt{A Z} + \sqrt[4]{A Z^2} + Z}{5}
\end{aligned}
$$

Limiting Behaviour

We now consider what happens as the number of dimensions increases.

$$
\begin{aligned}
M_\infty &= \lim_{n\to\infty} M_n \\
&= \lim_{n\to\infty} \left(\frac{1}{n+1} \cdot \sum_{k=0}^{n} \sqrt[n]{A^k Z^{n-k}} \right) \\
&= \frac{1}{1-0} \int_0^1 A^x Z^{1-x} \, \mathrm{d}x \\
&= \left[\frac{Z \cdot \mathrm{e}^{\ln A \cdot x - \ln Z \cdot x}}{\ln A - \ln Z} \right]_0^1 \\
&= \left[\frac{Z \cdot \left(\mathrm{e}^{(\ln A - \ln Z)}\right)^x}{\ln A - \ln Z} \right]_0^1 \\
&= \frac{1}{\ln A - \ln Z} \cdot \left[Z \left(\frac{A}{Z} \right)^x \right]_0^1 \\
&= \frac{1}{\ln A - \ln Z} \cdot \left(Z \left(\frac{A}{Z} \right)^1 - Z \left(\frac{A}{Z} \right)^0 \right) \\
&= \frac{1}{\ln A - \ln Z} \cdot (A - Z) \\
&= \frac{A - Z}{\ln A - \ln Z} = \frac{Z - A}{\ln Z - \ln A}
\end{aligned}
$$

That is, the limit is the logarithmic mean of the measures of the ‘ends’.